



ON LINE COURSE WARE

SUBJECT NAME: NUMERICAL METHODS

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TOTAL NO. OF LECTURES: 33

CREDIT: 3

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LESSION PLAN

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Module I: Numerical Method I

Error Analysis and Interpolation

LECTURE: 1.

Approximation in Numerical Computation

Introduction

The ultimate goal of the field of numerical analysis is to offersuitableapproaches for findingbeneficial solutions to mathematical problems and for removing useful information from available solutions which are not expressed in manageable forms. Such problems may be expressed, for example, in terms of an algebraic or transcendental equation, an ordinary or partial differential equation, or an integral equation, or in terms of a set of such equations etc.

Significant Figures:

The numbers 1,2,3,4,5,6,7,8,9 are all significant figures. But 0 is not constantly a significant figure. It is significant except when it is used to fix decimal point or to fill the place of unknown or discarded digits. As for example in the numbers 2407, 3500, 21780, 5.2307,zeros are significant. But in the numbers 0027,0.00057,0.12, 0.7800 zeros are not significant figures.

Errors:

In numerical calculation, sometimes methods are approximated and data are also inexact and in most cases result in numerical calculation will have some blunders. The errors are basically of two types of error are known as truncating error and other is computational error.

Truncation Error:

The truncation error is the error which is inherent in numerical data itself. Truncation error is also observed when initial data are not exact. In this case also we cannot be removed the error from the given data if the problem is solved correctly.

Computational Error:

The computational error arises during arithmetic calculation mainly due to the approximation of numbers. In arithmetic calculation, most numbers are used by taking finite number of digits. Again in the case of subtraction of nearly two equal numbers, multiplication or division of very small numbers, sometimes lack

of significant digits are observed. Accordingly computational error is again characterized into two types –first one is significant error and other is rounding off error.

LECTURE 2

Rounding off Error:

These types of error occur in arithmetic calculation when we round off numbers. When the given data are not exact, the data are used in calculation by taking their approximation only, then some errors occur and they propagate in arithmetic calculation these errors are known as rounding off error.

The general rule for rounding-off a number to n-significant figures are: Discard all digit to the right of the nth place, if the discarded number is less than half a unit in the nth place, leave the nth digit is unchanged; if the discarded number is greater than half a unit in the nth place, add 1(one) to the nth digit. If the discarded digit is exactly half a unit in the (n+1)th place, leave the nth digit unaltered if it is an even number, but increase it by 1 (one) if it is an odd number.

Example: Correct up to six significant figures:

- (i) 24.564932 becomes 24.5649 (ii) 24.564962 becomes 24.5650 (iii) 24.564652 becomes 24.5646
(iv) 24.564752 becomes 24.5648

Significant Error:

Significant error arises due to lack of significant digits when we subtraction of nearly two equal numbers, multiplication or division of small number by a second smaller number. As for example in the subtraction of two numbers 0.87251527 and 0.87251526123, if we take the values up to five decimal places, then serious error may be observed when the result is zero. Hence, significant error arises due to the loss of significant digits and not from the data.

Approximate Error:

Approximate error is defined as the difference between true value and approximate value. Let x_T and x_A denotes the true value and absolute value of any quantity or number then approximate error is denoted by E_A and is given by

$$E_A = x_T - x_A$$

Absolute error:

The absolute value of the approximate error is known as absolute error. Hence absolute error is given by

$$|E_A| = |x_T - x_A|$$

Relative error:

The ratio of absolute error and true value of a quantity is known as relative error. The relative error is given by

$$\frac{|E_A|}{x_T} \text{ and denoted by } E_R.$$

Percentage error:

Percentage error is the error which is 100 times the relative error. If E_P is the percentage error then

$$E_p = \frac{|E_A|}{x_T} \times 100$$

Theorem:

If a number which is correct upto n significant figures and the first significant digit is k, then the relative error $E_R \leq \frac{1}{k \cdot 10^{n-1}}$.

LECTURE 3

Fixed and floating-point arithmetic:

Any rounded number can be written in the standard form:

$$0.b_{-1}b_{-2}.....b_{-m} \times 10^n$$

Where n is an integer, positive, negative or zero and m is a positive integer and $b_{-1} \neq 0$. Then the rounded number is said to have m significant digits or figures. The above form of a rounded number is called the floating point representation of the given number.

Propagation of error:

We are alert that our given original data are exaggerated with round-off error which lies within the limit $\pm \frac{1}{2}$ in the last significant figure. Let us see how these original round-off errors move the successive differences in a difference table. If y_i denotes the exact value of $f(x_i)$, then its entered (by mistake or any how) value is $y_i - \varepsilon_i$, where ε_i is the analogous round-off error and the difference table continues as shown in the table below:

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	
x_0	$y_0 - \varepsilon_0$				
		$\Delta y_0 - \varepsilon_1 + \varepsilon_0$			
x_1	$y_1 - \varepsilon_1$		$\Delta^2 y_0 - \varepsilon_2 + 2\varepsilon_1 - \varepsilon_0$		
		$\Delta y_1 - \varepsilon_2 + \varepsilon_1$		$\Delta^3 y_0 - \varepsilon_3 + 3\varepsilon_2 - 3\varepsilon_1 + \varepsilon_0$	
x_2	$y_2 - \varepsilon_2$		$\Delta^2 y_1 - \varepsilon_3 + 2\varepsilon_2 - \varepsilon_1$	
		$\Delta y_2 - \varepsilon_3 + \varepsilon_2$		$\Delta^3 y_1 - \varepsilon_4 + 3\varepsilon_3 - 3\varepsilon_2 + \varepsilon_1$	
x_3	$y_3 - \varepsilon_3$		$\Delta^2 y_2 - \varepsilon_4 + 2\varepsilon_3 - \varepsilon_2$	
		$\Delta y_3 - \varepsilon_4 + \varepsilon_3$		$\Delta^3 y_2 - \varepsilon_5 + 3\varepsilon_4 - 3\varepsilon_3 + \varepsilon_2$	
x_4	$y_4 - \varepsilon_4$		$\Delta^2 y_3 - \varepsilon_5 + 2\varepsilon_4 - \varepsilon_3$		
		$\Delta y_4 - \varepsilon_5 + \varepsilon_4$			
x_5	$y_5 - \varepsilon_5$				

Ex1. Write down the approximate representation of 2/3 correct to four significant figures and then find

(a) Absolute error (b) relative error (c) relative percentage error.

Solution: The approximate representation of 2/3 correct up to four significant figures is 0.6667.

Therefore $X_T = 2/3$ and $X_A = 0.6667$

Absolute error $|E_A| = |X_T - X_A| = |2/3 - 0.6667| = 0.000033$

Relative error $E_R = E_A/X_T = 0.000033/(2/3) = 0.0000495 \approx 0.00005$

$$\text{Relative percentage error } E_P = E_R \times 100 = 0.005\%$$

LECTURE 4:

Interpolation:

Introduction:

Consider a single valued continuous function $y = f(x)$ defined over $[a, b]$ where $f(x)$ is known explicitly. It is easy to find the values of 'y' for a given set of values of 'x' in $[a, b]$. i.e., it is possible to get information of all the points (x, y) where $a \leq x \leq b$. But the converse is not so easy. That is, using only the points (x_0, y_0) , (x_1, y_1) , ..., (x_n, y_n) where $a \leq x_i \leq b$ $i = 0, 1, 2, \dots, n$, it is not so easy to find the relation between x and y in the form $y = f(x)$ explicitly. That is one of the problems we face in numerical differentiation or integration. Now we have first to find a simpler function, say $g(x)$, such that $f(x)$ and $g(x)$ agree at the given set of points and accept the value of $g(x)$ as the required value of $f(x)$ at some point x in between a and b . Such a process is called interpolation.

In linear interpolation, we are given with two pivotal values $f_0 = f(x_0)$ and $f_1 = f(x_1)$, and we approximate the curve of f by a chord (straight line) P_1 passing through the points (x_0, f_0) and (x_1, f_1) . Hence the approximate value $f(x)$ at the intermediate point $x = x_0 + rh$ is given by the linear interpolation formula

$$f(x) \approx P_1(x) = f_0 + r(f_1 - f_0) = f_0 + r\Delta f_0$$

$$\text{Where } r = \frac{x - x_0}{h} \text{ and } 0 \leq r \leq 1.$$

LECTURE 5:

Forward Differences:

Let us consider a function $y = f(x)$ defined for $(n+1)$ distinct arguments of x say $x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, \dots, x_n = x_0 + nh$ in $[a, b]$ and the corresponding functional values are $y_0 = f(x_0), y_1 = f(x_1) = f(x_0 + h), y_2 = f(x_2) = f(x_0 + 2h), \dots, y_n = f(x_n) = f(x_0 + nh)$. The values of x i.e. $x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, \dots, x_n = x_0 + nh$ are called the arguments or nodes and the values of the function corresponding to the function are called entries.

The forward difference of $y = f(x)$ is denoted by

$$\Delta f(x) = f(x+h) - f(x), \text{ i.e.}$$

$$\Delta f(x_0) = f(x_0 + h) - f(x_0) = y_1 - y_0 = \Delta y_0$$

$$\Delta f(x_0 + h) = f(x_0 + 2h) - f(x_0 + h) = y_2 - y_1 = \Delta y_1$$

$$\Delta f(x_0 + 2h) = f(x_0 + 3h) - f(x_0 + 2h) = y_3 - y_2 = \Delta y_2$$

.....

The back ward difference of $y = f(x)$ is given by

$$\nabla f(x) = f(x) - f(x-h) \text{ i.e.}$$

$$\nabla f(x_0 + h) = f(x_0 + h) - f(x_0) = y_1 - y_0 = \nabla y_1$$

$$\nabla f(x_0 + 2h) = f(x_0 + 2h) - f(x_0 + h) = y_2 - y_1 = \nabla y_2$$

$$\nabla f(x_0 + 3h) = f(x_0 + 3h) - f(x_0 + 2h) = y_3 - y_2 = \nabla y_3 \quad \text{and so on}$$

.....

Example : Evaluate $f(15)$, given that $f(10) = 46, f(20) = 66$.

Here $x_0 = 10, x_1 = 20, h = x_1 - x_0 = 20 - 10 = 10$,

$f_0 = f(x_0) = 46$ and $f_1 = f(x_1) = 66$.

Now to calculate $f(15)$, take $x = 15$, so that where $r = \frac{x - x_0}{h}$

and hence

$$f(15) \approx P_1(15) = f_0 + r(f_1 - f_0) = 46 + 0.5(66 - 46) = 56$$

LECTURE 6:

Newton's Forward Difference Interpolation Formula:

Using Newton's forward difference interpolation formula we find the n degree polynomial P_n which approximates the function $f(x)$ in such a way that P_n and f agrees at $n+1$ equally spaced arguments, so that $P_n(x_0) = f_0, P_n(x_1) = f_1, \dots, P_n(x_n) = f_n$, where $f_1 = f(x_1), \dots, f_n = f(x_n)$ are the values of f in the table.

Newton's forward difference interpolation formula is

$$f(x) \approx P_n(x) = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \dots + \frac{r(r-1)\dots(r-n+1)}{n!}\Delta^n f_0$$

$$\text{where } x = x_0 + rh, r = \frac{x - x_0}{h}, 0 \leq r \leq n.$$

Example Using Newton's forward difference interpolation formula and the following table evaluate $f(15)$.

X	f(x)	Δf_0	$\Delta^2 f_0$	$\Delta^3 f_0$	$\Delta^4 f_0$
10	46				
20	66	20			
30	81	15	-5		
40	93	12	-3	2	
50	101	8	-4	-1	-3

Here $x = 15, x_0 = 10$,

$$x_1 = 20, h = x_1 - x_0 = 20 - 10 = 10,$$

$$r = (x - x_0)/h = (15 - 10)/10 = 0.5,$$

$$f_0 = 46, \Delta f_0 = 20, \Delta^2 f_0 = -5, \Delta^3 f_0 = 2, \Delta^4 f_0 = -3.$$

Substituting these values in the Newton's forward difference interpolation formula for

$$n = 4, \text{ we obtain } f(x) \approx P_n(x) = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \dots + \frac{r(r-1)\dots(r-n+1)}{n!}\Delta^n f_0$$

so that

$$f(15) = 56.8672, \text{ correct to 4 decimal places.}$$

LECTURE 7

Newton's Backward Difference Interpolation Formula:

Newton's backward difference interpolation formula is

$$f(x) \approx P_n(x) = f_n + r\nabla f_n + \frac{r(r+1)}{2!}\nabla^2 f_n + \dots + \frac{r(r+1)\dots(r+n-1)}{n!}\nabla^n f_n$$

$$\text{where } x = x_n + rh, r = \frac{x - x_n}{h} \text{ and } -n \leq r \leq 0.$$

Problem: For the following table of values, estimate $f(7.5)$, using Newton's backward difference interpolation formula.

x	F	∇f_n	$\nabla^2 f_n$	$\nabla^3 f_n$	$\nabla^4 f_n$
1	1				
2	8	7			
3	27	19	12		
4	64	37	18	6	0
5	125	61	24	6	0
6	216	91	30	6	0
7	343	127	36	6	0
8	512	169	42	6	0

Since the fourth and higher order derivatives are zero, so

$$f(x) \approx P_n(x) = f_n + r\nabla f_n + \frac{r(r+1)}{2!}\nabla^2 f_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 f_n$$

where $x = x_n + rh$, $r = \frac{x - x_n}{h}$ and $-n \leq r \leq 0$

$$f(7.5) = 512 + (-0.5)(169) + \frac{(-0.5)(-0.5+1)}{2!}(42) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}6$$

$$= 421.875$$

LECTURE 8:

Lagrange's Interpolation Formula:

If the function $f(x)$ is known for $(n+1)$ distinct values of x say x_0, x_1, \dots, x_n which are not equispaced then we used Lagrange's interpolation formula.

The Lagrange's interpolation formula is given by

$$f(x) \approx L(x) = w(x) \sum_{r=0}^n \frac{f(x_r)}{(x - x_r)w'(x_r)}$$

$$w(x) = (x - x_0)(x - x_1)(x - x_2) \dots \dots \dots (x - x_n)$$

$$w'(x_r) = (x_r - x_0)(x_r - x_1) \dots \dots (x_r - x_{r-1})(x_r - x_{r+1}) \dots \dots (x_r - x_n)$$

Example: Certain corresponding values of x and $\log_{10} x$ are

(300, 2.4771), (304, 2.4829), (305, 2.4843) and (307, 2.4871). Find $\log_{10} 301$.

$$\log_{10} 301 = \frac{(-3)(-4)(-6)}{(-4)(-5)(-7)}(2.4771) + \frac{(1)(-4)(-6)}{(4)(-1)(-3)}(2.4829) + \frac{(1)(-3)(-6)}{(5)(1)(-2)}(2.4843) + \frac{(1)(-3)(-4)}{(7)(3)(2)}(2.4871)$$

$$= 1.2739 + 4.9658 - 4.4717 + 0.7106$$

$$= 2.4786.$$

LECTURE 9:

Example. Find the value of y at x = 0 given some set of values (-2, 5), (1, 7), (3, 11), (7, 34)?

Solution:

Given the known values are,

$$x = 0 ; x_0 = -2 ; x_1 = 1 ; x_2 = 3 ; x_3 = 7 ; y_0 = 5 ; y_1 = 7 ; y_2 = 11 ; y_3 = 34$$

Using the interpolation formula,

$$y = (x-x_1)(x-x_2).....(x-x_n)(x_0-x_1)(x_0-x_2).....(x_0-x_n) y_0 + (x-x_0)(x-x_2).....(x-x_n)(x_1-x_0)(x_1-x_2).....(x_1-x_n) y_1 + \dots + (x-x_1)(x-x_1).....(x-x_{n-1})(x_n-x_0)(x_0-x_1).....(x_n-x_{n-1}) y_n$$

$$y = (0-1)(0-3)(0-7)(-2-1)(-2-3)(-2-7) \times 5 + (0+2)(0-3)(0-7)(1+2)(1-3)(1-7) \times 7 + (0+2)(0-1)(0-7)(3+2)(3-1)(3-7) \times 11 + (0+2)(0-1)(0-3)(7+2)(7-1)(7-3) \times 34$$

$$y = 2127 + 496 + -7720 + 5154$$

$$y = 1087180$$

Example. We wish to interpolate $f(x) = x^3$ over the range $1 \leq x \leq 3$, given these three points:

$$x_0 = 1 \quad f(x_0) = 1$$

$$x_1 = 2 \quad f(x_1) = 8$$

$$x_2 = 3 \quad f(x_2) = 27$$

The interpolating polynomial is:

$$L(x) = 1 \cdot \frac{x-2}{1-2} \cdot \frac{x-3}{1-3} + 8 \cdot \frac{x-1}{2-1} \cdot \frac{x-3}{2-3} + 27 \cdot \frac{x-1}{3-1} \cdot \frac{x-2}{3-2}$$

$$= 6x^2 - 11x + 6.$$

Module Number II:

Numerical Solution of Linear and Non-linear Equations

LECTURE 10:

Numerical Solution of A System Of Linear Equations (3I)

Introduction

A system of linear algebraic equations in n unknowns $x_i (i=1,2,\dots,n)$ can be expressed as follows

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\dots \dots \dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned} \quad \dots \dots \dots (1)$$

Here, coefficients a_{ij} and the constants b_i are given. A solution of (1) is a set of values x_i which satisfies (1). (1) is said to be homogeneous if all b_i are equal or zero, otherwise the system is called non-homogeneous. The system can be written as $AX=B$ where A is the $(n \times n)$ coefficient matrix, where we assume that A is non-singular. The system of equations can be solved in two ways 1) direct method and 2) iterative method. Through finite steps of elementary arithmetic operations we obtain the solution in direct method where as in

iterative method we assume an initial approximation and improve the solution by iteration. Here we will discuss the different methods. In direct method the following will be described

1) **Gauss Elimination**

2) **LU factorization method**

and

3) **Gauss-Seidel method** which is a iterative method will also be described.

LECTURE 12:

Gauss-Elimination Method

In this method, the system is converted to an equivalent upper triangular system by a systematic elimination procedure.

To discuss about the method let us consider the following system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned} \quad \dots\dots\dots(2)$$

Let us take $a_{11} \neq 0$. Now multiplying the 1st equation of (2) by a_{i1}/a_{11} and subtract from i th equations we obtain

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{22}^{(1)}x_2 + \dots + a_{2n}^{(1)}x_n &= b_2^{(1)} \\ &\dots\dots\dots \\ a_{n2}^{(1)}x_2 + \dots + a_{nn}^{(1)}x_n &= b_n^{(1)} \end{aligned} \quad \dots\dots\dots(3)$$

where $a_{ij}^{(1)} = a_{ij} - a_{1j} \cdot a_{i1}/a_{11}$

$b_i^{(1)} = b_i - b_1 \cdot a_{i1}/a_{11}$ where $(i, j=2, \dots, n)$

the numbers a_{i1}/a_{11} are called row multipliers, the first equation of the system (3) contain x_1 while the other $(n-1)$ equations are independent of x_1 .

In the next step we take $a_{22}^{(1)} \neq 0$. Multiplying the 2nd equation of (3) by $a_{i2}^{(1)}/a_{22}^{(1)}$ (where $i=3, 4, \dots, n$) we obtain

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{22}^{(1)}x_2 + \dots + a_{2n}^{(1)}x_n &= b_2^{(1)} \\ a_{33}^{(2)}x_3 + \dots + a_{3n}^{(2)}x_n &= b_3^{(2)} \\ &\dots\dots\dots \\ a_{n3}^{(2)}x_3 + \dots + a_{nn}^{(2)}x_n &= b_n^{(2)} \end{aligned} \quad \dots\dots\dots(4)$$

where $a_{ij}^{(2)} = a_{ij}^{(1)} - a_{2j}^{(1)} \cdot a_{i2}^{(1)}/a_{22}^{(1)}$

$b_i^{(2)} = b_i^{(1)} - b_2^{(1)} \cdot a_{i2}^{(1)}/a_{22}^{(1)}$ where $(i, j=3, \dots, n)$

Here $a_{i2}^{(1)}/a_{22}^{(1)}$ is also row multiplier. Proceeding as above we can reduce the given system to an upper triangular system as follows

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{22}^{(1)}x_2 + \dots + a_{2n}^{(1)}x_n &= b_2^{(1)} \\ a_{33}^{(2)}x_3 + \dots + a_{3n}^{(2)}x_n &= b_3^{(2)} \\ &\dots\dots\dots \\ a_{nn}^{(n-1)}x_n &= b_n^{(n-1)} \end{aligned} \quad \dots\dots\dots(5)$$

The co-efficient of leading terms in (5) i.e. $a_{11}, a_{22}^{(1)}, \dots$ is called pivotal elements and corresponding equations are pivotal equations. At the last stage by back substitution the solution occurs.

Example 1:

Solve the following system of equations by Gauss Elimination method

$$\begin{aligned} x + 2y + z &= 0 \\ 2x + 2y + 3z &= 3 \\ -x - 3y &= 2 \end{aligned}$$

Solution: To solve the given system of linear equations by using Gauss Elimination method, let us consider augmented matrix to get upper triangular matrix by row operation

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 2 & 3 & 3 \\ -1 & -3 & 0 & 2 \end{array} \right] \xrightarrow{\substack{R_2' \leftarrow R_2 - 2R_1 \\ R_3' \leftarrow R_3 + R_1}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & -1 & 1 & 2 \end{array} \right] \xrightarrow{R_3' \leftarrow R_3 - \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array} \right]$$

Thus the given system of equations reduces to the following upper triangular system as

$$\begin{aligned} x+2y+z &= 0 \\ 2y+z &= 3 \\ \frac{z}{2} &= \frac{1}{2} \end{aligned}$$

By back substitution we obtain the required solution as $z=1, y=-1, x=1$.

Hence the required solution is $x=1, y=-1, z=1$.

LECTURE 13:

LU-FACTORIZATION METHOD:

This method is also said as triangular decomposition method. The method based on the fact that every square matrix can be expressed as the product of a lower and upper triangular matrix provided all the principal minors of the given square matrix $A=(a_{ij})$ are nonsingular i.e $a_{11} \neq 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \dots, \det A \neq 0$.

Moreover, if the matrix can be factorized, then it is unique. Assume that it is possible to decompose the coefficient matrix A of the given system of equation (2) and is expressible as the product of a lower triangular matrix L and an upper triangular matrix U so that

$$A=LU \quad \dots\dots(6)$$

Where

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \dots & \dots & u_{1n} \\ 0 & u_{22} & u_{23} & & & \cdot \\ 0 & 0 & u_{33} & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & u_{nn} \end{bmatrix} \quad L = \begin{bmatrix} l_{11} & 0 & 0 & \dots & \dots & 0 \\ l_{21} & l_{22} & 0 & & & \cdot \\ l_{31} & l_{32} & l_{33} & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & & 0 \\ l_{n1} & \cdot & \cdot & \cdot & \cdot & l_{nn} \end{bmatrix} \quad \dots\dots\dots(7)$$

Hence the system of equations $AX=b$ become $LUX=b$ \dots\dots\dots(8)

Putting $UX=Y$ in (8) we get

$$LY=b \quad \dots\dots\dots(9)$$

Where $Y=(y_1, y_2, \dots, y_n)^T$

Thus using forward substitution, the unknowns y_1, y_2, \dots, y_n are determined from (9) and then the unknowns x_1, x_2, \dots, x_n are obtained from (8) by backward substitution.

Now we consider a system of three equations with three unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \quad \dots\dots(10)$$

Here the coefficient matrix is $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ can be written as

$$A=LU$$

$$\text{Where } L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Thus we have } \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Leading to

$$l_{11}=a_{11}, l_{21}=a_{21}, l_{31}=a_{31}$$

$$l_{11}u_{12}=a_{12}, l_{11}u_{13}=a_{13} \text{ which implies } u_{12}=a_{12}/l_{11}, u_{13}=a_{13}/l_{11},$$

$$l_{21}u_{12}+u_{22}=a_{22} \text{ which implies } l_{22}=a_{22}-l_{21}u_{12}=a_{22}-a_{21}.a_{12}/a_{11}$$

$$l_{31}u_{12}+l_{32}=a_{32} \text{ which implies } l_{32}=a_{32}-a_{31}.a_{12}/a_{11}$$

$$l_{21}u_{13}+l_{22}u_{23}=a_{23} \text{ which implies } u_{23}=a_{23}-l_{21}u_{13}/l_{22}$$

$$\text{and } l_{31}u_{13}+l_{32}u_{23}+l_{33}=a_{33} \text{ which implies } l_{33}=a_{33}-l_{31}u_{13}-l_{32}u_{23}.$$

Thus we obtained values of l_{11}, l_{21}, \dots and u_{12}, u_{13}, \dots gives the matrices L and U.

Example 2:

Solve the following system

$$3x_1+2x_2-4x_3=12$$

$$-x_1+5x_2+2x_3=1$$

$$2x_1-3x_2+4x_3=-3$$

By LU-Factorization method.

Solution: Let, $A=LU$, therefore $LU=A$, then we can write as

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -4 \\ -1 & 5 & 2 \\ 2 & -3 & 4 \end{bmatrix}$$

Or,
$$\begin{bmatrix} l_{11} & l_{11}.u_{12} & l_{11}.u_{13} \\ l_{21} & l_{21}.u_{12} + l_{22} & l_{21}.u_{13} + l_{22}.u_{23} \\ l_{31} & l_{31}.u_{12} + l_{32} & l_{31}.u_{13} + l_{32}.u_{23} + l_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & -4 \\ -1 & 5 & 2 \\ 2 & -3 & 4 \end{bmatrix}$$

Therefore $l_{11}=3$

$$l_{11}u_{12}=2 \rightarrow u_{12}=2/3$$

$$l_{11}.u_{12}=-4 \rightarrow u_{13}=-4/3$$

$$l_{21}=-1$$

$$l_{21}u_{12}+l_{22}=5 \rightarrow l_{22}=17/3$$

$$l_{21}u_{13}+l_{22}u_{23}=2 \rightarrow u_{23}=2/17$$

$$l_{31}=2$$

$$l_{31}u_{12}+l_{32}=-3 \rightarrow l_{32}=-13/3$$

$$l_{31}u_{13}+l_{32}u_{23}+l_{33}=4 \rightarrow l_{33}=366/51$$

$$\text{Thus } \begin{bmatrix} 3 & 0 & 0 \\ -1 & 17/3 & 0 \\ 2 & -13/3 & 366/51 \end{bmatrix} \begin{bmatrix} 1 & 2/3 & -4/3 \\ 0 & 1 & 2/17 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -4 \\ -1 & 5 & 2 \\ 2 & -3 & 4 \end{bmatrix}$$

From (9) we get

$$\begin{bmatrix} 3 & 0 & 0 \\ -1 & 17/3 & 0 \\ 2 & -13/3 & 366/51 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 1 \\ -3 \end{bmatrix}$$

Solving we obtain

$$y_1=4$$

$$y_2=15/17$$

$$y_3=-1$$

Again from (6) we get

$$\begin{bmatrix} 1 & 2/3 & -4/3 \\ 0 & 1 & 2/17 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 15/17 \\ -1 \end{bmatrix}$$

Solving we get $x_1=2$, $x_2=1$ and $x_3=-1$.

LECTURE 14:

Gauss-Seidel Method:

This method is an improvised form of Gauss Jacobi method, since the improved values of x_i are used here in each iteration rather than the values of the previous iteration and thus this method is also called Method of Successive Displacements.

To discuss the method, we rewrite the system of equations (1) as

$$X_1 = (b_1 - a_{12}X_2 - a_{13}X_3 - \dots - a_{1n}X_n)/a_{11}$$

$$x_2 = (b_2 - a_{21}X_1 - a_{23}X_3 - \dots - a_{2n}X_n)/a_{22}$$

.....

$$X_n = (b_n - a_{n1}X_1 - a_{n2}X_2 - \dots - a_{nn-1}X_{n-1})/a_{nn} \quad \dots\dots(11)$$

Provided $a_{ii} \neq 0 (i=1, 2, \dots, n)$

To solve the equation (11) suppose $x_1^{(0)}, \dots, x_n^{(0)}$ be the initial approximation of the equation (1). Now substituting those initial values in the RHS of the first equation of (11) and get the first approximation of x_1 as

$$X_1^{(1)} = (b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)} - \dots - a_{1n}x_n^{(0)})/a_{11}$$

In the second equation of (11) we substitute the improved value $x_1^{(1)}$ and initial values $x_3^{(0)}, x_4^{(0)}, \dots, x_n^{(0)}$ and obtain the first approximation of x_2 as

$$x_2^{(1)} = (b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)} - \dots - a_{2n}x_n^{(0)})/a_{22}$$

Next we substitute in the third equation of (11) the improved values of $x_1^{(1)}, x_2^{(1)}$ and the initial values of $x_4^{(0)}, x_5^{(0)}, \dots, x_n^{(0)}$ to obtain the first approximation of x_3 as

$$x_3^{(1)} = (b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)} - a_{34}x_4^{(0)} - \dots - a_{3n}x_n^{(0)})/a_{33}$$

Proceeding in such manner we finally obtain

$$X_n^{(1)} = (b_n - a_{n1}x_1^{(1)} - a_{n2}x_2^{(1)} - \dots - a_{nn-1}x_{n-1}^{(1)})/a_{nn}$$

Thus the end of the first stage of iteration, we get first approximation $x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(1)}$ to the solution x_1, x_2, \dots, x_n .

Now if $x_i^{(k)} (k=0, 1, 2, \dots, n)$ be the k th approximation to the solutions of $x_i (i=1, 2, \dots, n)$ then the $(k+1)$ th approximation $x_i^{(k+1)}$ of x_i are given by

$$X_1^{(k+1)} = (b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \dots - a_{1n}x_n^{(k)})/a_{11}$$

$$x_2^{(k+1)} = (b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)} - \dots - a_{2n}x_n^{(k)})/a_{22}$$

.....

$$X_n^{(k+1)} = (b_n - a_{n1}x_1^{(k+1)} - a_{n2}x_2^{(k+1)} - \dots - a_{nn-1}x_{n-1}^{(k+1)})/a_{nn} \quad \dots\dots(11)$$

The process is continued until we obtain the solution x_1, x_2, \dots, x_n with sufficient degree of accuracy.

The sequence $\{x_i^{(n)}\}$ generated from (11) can be shown to be convergent to the solution $\{x_i^*\}$ if

$$|a_{ii}| > \sum_{j=1, j \neq i}^n |a_{ij}|, (i=1, 2, \dots, n) \quad \dots\dots(12)$$

Hence the Gauss-Seidel iteration is convergent if the system of equations (1) is strictly diagonally dominant.

Example 3:

Solve the following system of equation by Gauss-Seidel method

$$20x_1 + 5x_2 - 2x_3 = 14$$

$$3x_1 + 10x_2 + x_3 = 17$$

$$x_1 - 4x_2 + 10x_3 = 23$$

Correct upto 4 decimal places.

Solution:

The system is obviously diagonally dominant.

We write the corresponding iteration formulae as:

$$X_1^{(k+1)} = \frac{1}{20}[14 - 5x_2^{(k)} + 2x_3^{(k)}]$$

$$X_2^{(k+1)} = \frac{1}{10}[17 - 3x_1^{(k+1)} - x_3^{(k)}]$$

$$X_3^{(k+1)} = \frac{1}{10}[23 - x_1^{(k+1)} + 4x_2^{(k+1)}]$$

Computation table is given below

k	$X_1^{(k)}$	$X_2^{(k)}$	$X_3^{(k)}$
0	0	0	0
1	0.7	1.5	3.5
2	0.68	1.15	2.69
3	0.682	1.226	2.722
4	0.6667	1.2278	2.7244
5	0.6665	1.2279	2.7246
6	0.6655	1.2279	2.7246

Thus $x_1=0.6655$, $x_2=1.2279$, $x_3=2.7246$ correct up to 4 decimal places.

LECTURE 15:

Successive Over Relaxation Method

Any second order Elliptic Equation e.g. Laplace Equation, while solving by FD method, always reduces to a equation containing $u_{i,j}$, $u_{i-1,j}$, $u_{i+1,j}$ and $u_{i,j+1}$ which for different values of (i, j) may be converted to a system of equations. Rewriting equation again:

$$U_{i+1,j}+u_{i-1,j}+u_{i,j-1}+u_{i,j+1}-4u_{i,j}=(\Delta x)^2 f(i\Delta x, j\Delta y)$$

Putting $j=1$, $i=1,2,\dots,N$, then $j=2$, $i=1,2,\dots,N$ and in the same manner $j=M$, $i=1,2,\dots,N$, thus finally we obtain a system of $M \times N$ equations which are solved by any convenient method to get $u_{i,j}$ at all nodes. This is one of the implicit scheme but however sometimes the size of the system becomes so large that it is unmanageable. Hence, this method is not an efficient way for solving Elliptic PDE.

One of the techniques used for this purpose is the Successive over relaxation (SOR) method.

To explain its implementation, we can write

$$U_{i,j}^{n+1}=u_{i,j}^n+\frac{w}{4}[(\Delta x)^2 f_{i,j}-u_{i-1,j}^{n+1}-u_{i,j-1}^{n+1}-u_{i,j+1}^n-u_{i+1,j}^n-4u_{i,j}^n]$$

$$\text{Thus } U_{i,j}^{n+1}=w(\text{R.H.S of Gauss-Seidel method})-(w-a)u_{i,j}^n$$

w is called as a relaxation parameter. It is verified that for over relaxation method, $1 < w < 2$.

For a rectangular domain, a reasonable estimate of ω can be taken as the small root of the equation

$$\left(\cos \frac{\pi}{M} + \cos \frac{\pi}{N}\right)^2 w^2 - 16w + 16 = 0$$

where M and N are number of sub-domain in each side.

$$\text{Therefore } w = \frac{4}{2 + \sqrt{4 - k^2}}$$

$$\text{where } k = \left(\cos \frac{\pi}{M} + \cos \frac{\pi}{N}\right)$$

The following table shows the value of ω for different M & N

M=N	W2
2	1.000
3	1.072
5	1.260
10	1.528
20	1.729
∞	2.000

It is observed that usually $W_{opt}=1.5$

For large M and N, one can approximate w by

$$W_{\text{opt}} = 2 - \sqrt{2\pi \left(\frac{1}{m^2} + \frac{1}{n^2} \right)^{0.5}} \approx 2 \left(1 - \frac{\pi}{M} \right), M=N.$$

EXAMPLE

Solve $\nabla^2 u = 0$ subject to Dirichlet conditions in the domain $0 \leq x \leq 0.75$, $0 \leq y \leq 0.75$ with $\Delta x = \Delta y = 0.25$ subject to the boundary conditions: $u(0,y) = 0$, $u(0.75,y) = 10$ and $u(x,0) = 0$, $u(x,0.75) = 10$ by SOR Method.

SOLUTION

$k = \left(\cos \frac{\pi}{M} + \cos \frac{\pi}{N} \right)$, M, N = no. of sub – divisions. Here $M = 3$, $N = 3$.

The equations in the iterative form can be written as follows:

$$W = \frac{4}{2 + \sqrt{4 - k^2}}$$

Therefore $w = 1.072$.

(Start with assumed values $u_{i,j}^0 = 0$)

$$U_{1,1}^{n+1} = 1.072 \left[\frac{1}{2} (u_{2,1}^n + 5) \right] - 0.072 u_{1,1}^n$$

$$U_{2,1}^{n+1} = 1.072 \left[\frac{1}{4} (u_{2,1}^n + u_{2,2}^n + 10) \right] - 0.072 u_{2,1}^n$$

$$U_{2,2}^{n+1} = 1.072 \left[\frac{1}{2} (u_{2,1}^{n+1} + 10) \right] - 0.072 u_{2,2}^n$$

The values so computed are given in the following table:

	$U_{1,1}$	$U_{2,1}$	$U_{2,2}$
n=0	0	2.68	6.70
n=1	1.4365	4.6676	7.3794
n=2	2.3984	4.9644	7.4896
n=3	2.4882	4.9844	7.4817
n=4	2.4988	4.9914	7.4917

Hence solution correct to 2d is: $u_{1,1} = 2.50$; $u_{2,1} = 5.00$; $u_{2,2} = 7.50$

Thus SOR Method gives faster convergence.

LECTURE 16:

Numerical Solutions of Algebraic And Transcendental Equations (3L)

BISECTION METHOD:

Working principle: It is an direct method and is based on a well known theorem which states that if $f(x)$ be a continuous function in a closed interval $[a,b]$ and $f(a)$ and $f(b)$ are of opposite signs [i.e. $f(a)f(b) < 0$], then there exists at least one real root of the equation $f(x) = 0$, between a and b. If further $f'(x)$ exists and $f'(x)$ maintains same sign in $[a,b]$, i.e., $f(x)$ strictly monotonic, then there is only one root of $f(x) = 0$, in (a,b) .

The computational scheme of the method is find an interval $[a_0, b_0]$ where $f(a_0)f(b_0) < 0$ and $f'(x)$ maintains same sign. To generate the sequence $\{x_n\}$, let us put $x_0 = a_0$ or b_0 and $x_1 = (a_0 + b_0)/2$ and find $f(x_1)$. If $f(a_0)$ and $f(x_1)$ are of opposite in signs, then we can set $a_1 = a_0$ and $b_1 = x_1$ so that $[a_1, b_1] = [a_0, x_1]$. On the other hand, if

$f(x_1)$ and $f(b_0)$ are of opposite in signs the we can set $a_1=x_1$ and $b_1=b_0$, i.e. $[a_1,b_1] = [x_1,b_0]$. Thus we see that there is a real root α in $[a_1,b_1]$ in either case. Similarly next we can set $x_2=(a_1+b_1)/2$, Repeating the above process we get $x_{n+1} = (a_n+ b_n)/2$.

Advantage of bisection method:

- The iteration using bisection method always produces a root, since the method brackets the root between two values.
- As iterations are conducted, the length of the interval gets halved. Thus this method surely convergent and converge to a real root (i.e. the solution) of the equation.
- The Bisection Method is easily programmable in any programming language.

Disadvantage of bisection method:

- The convergence of the bisection method is slow as it is simply based on halving the interval.
- Bisection method cannot be applied over an interval where there is a discontinuity.
- Bisection method cannot be applied over an interval where the function takes always values of the same sign.
- The method fails to determine complex roots.
- If one of the initial guesses $[a_0, b_0]$ is closer to the exact solution, it will take larger number of iterations to reach the root.

Example.1. Find a real root of the equation $x^3-3x + 1.06 = 0$ using bisection method correct to three decimal places.

Solution: The given equation is $x^3-3x + 1.06 = 0$, Let $f(x) = x^3-3x + 1.06$.

Now $f(0) = 1.06$ (+) ive, $f(1) = -0.9400$, thus a (+) real root lies in the interval $(0,1)$.

Hence $x_1 = (0+1)/2 = 0.5$ and $f(0.5) = -0.3150$, the root lies in the interval $(0,0.5)$. Bisection method tabulated as:

n	$a_n(+)$	$b_n(-)$	$x_n = (a_n+b_n)/2$	$f(x_n)$
0	0.0000	1.0000	0.5000	-0.3150
1	0.0000	0.5000	0.2500	+0.3256
2	0.2500	0.5000	0.3750	-0.0123
3	0.2500	0.3750	0.3125	+0.1530
4	0.3125	0.3750	0.3438	+0.0694
5	0.3438	0.3750	0.3594	+0.0283
6	0.3594	0.3750	0.3672	+0.0079
7	0.3672	0.3750	0.3711	-0.0022
8	0.3672	0.3711	0.3691	+0.0029
9	0.3691	0.3711	0.3701	+0.0003
10	0.3701	0.3711	0.3706	-0.0009
11	0.3701	0.3706	0.3704	-0.0003
12	0.3701	0.3704	0.3702	+0.0000
13	0.3702	0.3704	0.3703	-0.0001

Hence a positive real root of the given equation is 0.370 in $(0,1)$ correct to 3 decimal places.

Example.2. Find a smallest real root of the equation $e^x = 4\sin(x)$ using bisection method correct to three decimal places.

Solution: The given equation is $e^x = 4\sin(x)$ i.e. $4\sin(x) - e^x = 0$, Let $f(x) = 4\sin(x) - e^x$

Now $f(0) = -1.0$ (-)ive, $f(1) = 0.6476$ (+)ve, thus a (+) real root lies in the interval $(0,1)$.

Hence $x_1 = (0+1)/2 = 0.5$ and $f(0.5) = 0.26898$, the root lies in the interval $(0,0.5)$. Bisection method tabulated as:

n	a_n	b_n	$x_n=(a_n+b_n)/2$	$f(x_n)$
0	0.0000	1.0000	0.5000	+0.2690
1	0.0000	0.5000	0.2500	-0.2944
2	0.2500	0.5000	0.3750	+0.0101
3	0.2500	0.3750	0.3125	-0.1371
4	0.3125	0.3750	0.3438	-0.0621
5	0.3438	0.3750	0.3594	-0.0257
6	0.3594	0.3750	0.3672	-0.0077
7	0.3672	0.3750	0.3711	+0.0012
8	0.3672	0.3711	0.3691	-0.0032
9	0.3691	0.3711	0.3701	-0.0010
10	0.3701	0.3711	0.3706	+0.0001
11	0.3701	0.3706	0.3704	-0.0004
12	0.3704	0.3706	0.3705	-0.0002
13	0.3705	0.3706	0.3705	-0.0000

Hence a smallest positive real root of the given equation is 0.371 in (0,1) correct to 3 decimal places.

LECTURE 17:

REGULA-FALSI METHOD (Method of False Position / Method of Chords):

Regula-Falsi Method (Method of False Position) for finding a real root of an equation $f(x)=0$ is an oldest method. In this method we need approximate location of the root in the given interval (a,b) .

This method is also based on the intermediate value theorem as like bisection method. Like bisection method, we choose two points a_n and b_n such that $f(a_n)$ and $f(b_n)$ are of opposite signs (i.e., $f(a_n)f(b_n) < 0$).

Then, intermediate value theorem suggests that a zero of $f(x)$ lies in between a_n and b_n , if f is a continuous function.

Given a function $f(x)$ continuous on an interval $[a_0, b_0]$ and satisfying $f(a_0)f(b_0) < 0$.

For $n = 0, 1, 2, \dots$ until termination do:

Compute $x_1 = a_0 - \frac{f(a_0)(b_0 - a_0)}{f(b_0) - f(a_0)}$. Then compute $f(x_1)$.

If $f(a_0)f(x_1) < 0$, then we set $a_1 = a_0$ & $b_1 = x_1$ and compute x_2 and hence $f(x_2)$.

If $f(a_0)f(x_1) > 0$, then we set $a_1 = x_1$ & $b_1 = b_0$ and compute x_2 and hence $f(x_2)$.

Maintaining the same procedure

$$x_{n+1} = a_n - \frac{f(a_n)(b_n - a_n)}{f(b_n) - f(a_n)}. \text{ If } f(x_{n+1}) = 0 \text{ then } x_{n+1} \text{ is the root of the given equation.}$$

Advantage and disadvantage of Regula Falsi Method:

1. The formula is very simple for computation'
2. The method is evidently convergent on the other hand the disadvantage of the method is very slow.
3. The interval in which the root lies must be very small.

Example.1. Find a real root of the equation $x^2 - \sin(x) = 0$ using Regula-Falsi method correct to four(4) decimal places.

Solution: The given equation is $x^2 - \sin(x) = 0$. Let $f(x) = x^2 - \sin(x)$.

Now $f(0) = 0$, $f(1) = 0.1585$ and $f(0.5) = -0.2294$. Thus a (+) real root lies in the interval (0.5,1).

Let us consider $x_0 = 0.5$ and $x_1 = 1.0$. Then by Regula-Falsi iterative formula

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n), \quad n = 1, 2, 3, \dots \quad \text{This is we can write in the form as}$$

$$x_{n+1} = x_n - h_n, \quad \text{where } h_n = \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n), \quad n = 1, 2, 3, \dots$$

$$\text{Therefore } x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1) = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1) = 1 - \frac{(1 - 0.5)}{f(1) - f(0.5)} f(1)$$

$$x_2 = 1 - \frac{(1 - 0.5)}{0.1585 - (-0.2294)} (0.1585) = 1 - \frac{0.5}{0.3880} (0.1585) = 0.7957$$

Since $f(0.7957) = -0.0812$,

Regula-Falsi method tabulated as:

NI	x_{n-1}	x_n	$f(x_{n-1})$	$f(x_n)$	h_n	x_{n+1}	$f(x_{n+1})$
0	0.50000	1.00000	-0.22943	0.15853	0.20431	0.79569	-0.08123
1	0.79569	1.00000	-0.08123	0.15853	0.13509	0.86491	-0.01297
2	0.86491	1.00000	-0.01297	0.15853	0.12488	0.87512	-0.00178
3	0.87512	1.00000	-0.00178	0.15853	0.12349	0.87651	-0.00024
4	0.87651	1.00000	-0.00024	0.15853	0.12330	0.87670	-0.00003
5	0.87670	1.00000	-0.00003	0.15853	0.12328	0.87672	-0.00000

Hence a real root of the given equation is 0.8767 correct to four(4) decimal places.

Example.2. Find a real root of the equation $2x - \log_{10}x - 7 = 0$ using Regula-Falsi method correct to three decimal places in the interval (3,4).

Solution: The given equation is $2x - \log_{10}x - 7 = 0$. Let $f(x) = 2x - \log_{10}x - 7$.

Now $f(0) = 0$, $f(1) = 0.1585$ and $f(0.5) = -0.2294$. Thus a (+) real root lies in the interval (0.5,1).

Let us consider $x_0 = 0.5$ and $x_1 = 1.0$. Then by Regula-Falsi iterative formula

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n), \quad n = 1, 2, 3, \dots \quad \text{This is we can write in the form as}$$

$$x_{n+1} = x_n - h_n, \quad \text{where } h_n = \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n), \quad n = 1, 2, 3, \dots$$

$$\text{Therefore, } x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1) = 4 - \frac{(4 - 3)}{f(4) - f(3)} f(4) = 4 - \frac{1}{f(4) - f(3)} f(4)$$

$$x_2 = 4 - \frac{1}{(0.39794) - (-1.47712)} (0.39794) = 4 - 0.21223 = 3.78777$$

Since $f(3.78777) = -0.00284$,

Regula-Falsi method tabulated as:

NI	x_{n-1}	x_n	$f(x_{n-1})$	$f(x_n)$	h_n	x_{n+1}	$f(x_{n+1})$
0	3.00000	4.00000	-1.47712	0.39794	0.21223	3.78777	-0.00284
1	3.78777	4.00000	-0.00284	0.39794	0.21072	3.78928	-0.00000
2	3.78928	4.00000	-0.00000	0.39794	0.21072	3.78928	-0.00000

Hence a real root of the given equation is 3.789 correct to three(3) decimal places.

LECTURE 18:

Secant Method:

The Newton-Raphson method needs the estimate of two functions (the function and its derivative) for each iteration. If they are complex expressions, it will take large amount of effort to handle the calculations. Hence it is necessary to have a method that converges as fast as Newton's method yet involves only the estimate of the function. Let x_0 and x_1 are two initial approximations for the roots of $f(x) = 0$ and $f(x_0)$ & $f(x_1)$ respectively, are their function values. If x_2 is the point of intersection of x-axis and the line-joining the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ then x_2 is closer to the desired root i.e. the root of the given equation, than x_0 and x_1 . The equation relating x_0 , x_1 and x_2 is found by considering the slope 'm' of the line.

$$m = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{0 - f(x_1)}{x_2 - x_1}$$

$$\Rightarrow x_2 - x_1 = \frac{-f(x_1) * (x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$\Rightarrow x_2 = x_1 - \frac{f(x_1) * (x_1 - x_0)}{f(x_1) - f(x_0)}$$

or in general the iterative procedure can be written as

$$x_{i+1} = x_i - \frac{f(x_i) * (x_i - x_{i-1})}{f(x_i) - f(x_{i-1})} \quad i = 1, 2, 3, \dots$$

Ex. Solve equation $f(x) = 10^x + x - 4 = 0$ correct to six significant figures by the secant method.

Solution:

Take $x_0 = 0.5$, $x_1 = 0.6$ and given $f(x) = 10^x + x - 4$.

Computation:

Therefore required root of the given equation is **0.539179**

LECTURE 19:

Newton-Raphson Method (Method of Tangents):

Working principle: The Newton-Raphson method is an iterative method and is used to find isolated roots of an equation $f(x)=0$. The object of this method is to correct the approximate root x_0 (say) successively to its exact value α . Initially a crude approximation small interval (a_0, b_0) is found out in which only one root α (say) of $f(x) = 0$ lies.

If $f'(x)$ is a simple expression, the real root of $f(x) = 0$ can be computed very easily by this method, provided $f'(x) \neq 0$.

Firstly find a crude approximation $[a_0, b_0]$ where $f(a_0).f(b_0) < 0$ and $f'(x)$ maintains same sign. Take x_0 ($a_0 \leq x_0 \leq b_0$) as an initial approximation of the root α of the equation $f(x) = 0$. The working formula of Newton-Raphson method is:

$$x_{n+1} = x_n + h_n \quad \text{where } h_n = -f(x_n)/f'(x_n) \quad (n= 0,1,2,3,\dots) \text{ and } f'(x_n) \neq 0 \quad (n= 0,1,2,3,\dots).$$

The method is also known as tangent method.

Advantages:

Since the rate of Convergence of the method is quadratic the method converges more rapidly. If the function is nearly vertical when it crosses the x-axis, the correct value of the root can be found with great rapidity and very little execution.

Disadvantages:

It is evident from the formula that larger derivative $f'(x)$ the smaller is the correction which must be applied to get the correct value of the root.

If $f'(x)$ is very small in the neighborhood of the root the values of $h = -\frac{f(x_k)}{f'(x_k)}$ would be large and the

computation of the root by this method would be a slow process or might even fail together.

n	x_n	$y_n = f(x_n)$	$h_n = -\frac{f(x_i) * (x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$	$x_{n+1} = x_n + h_n$
0	0.5	-0.3377		
1	0.6	0.5811	-0.6	0.54
2	0.54	0.007369	-0.008	0.5392
3	0.5392	0.0001873	-0.000021	0.539179
4	0.539179	-0.000001095	0.000000122	0.5391791
5	0.5391791	-0.000000198	0.000000022	0.539179122

The sequence converges provided that the initial approximation is chosen sufficiently close to the root otherwise the sequence may diverge.

Example.1. Find a real root of the equation $e^x = 2x + 1$ correct to four(4) decimal places using Newton-Raphson method near $x=1$.

Solution: The given equation is $e^x = 2x + 1$ i.e. $e^x - 2x - 1 = 0$.
Let $f(x) = e^x - 2x - 1$, $\therefore f'(x) = e^x - 2$

Then by Newton-Raphson iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad [f'(x_n) \neq 0], \quad n = 0, 1, 2, 3, \dots$$

This is we can write in the form as

$$x_{n+1} = x_n - h_n, \quad \text{where } h_n = \frac{f(x_n)}{f'(x_n)} \quad [f'(x_n) \neq 0], \quad n = 0, 1, 2, 3, \dots$$

Let us consider $x_0 = 1$

Therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{-0.2817}{0.7183} = 1.3922$$

Newton-Raphson iterative method tabulated as:

NI	x_n	$f(x_n)$	$f'(x_n)$	$h=f(x_n)/f'(x_n)$	x_{n+1}
0	1.00000	-0.28172	0.71828	-0.39221	1.39221
1	1.39221	0.23932	2.02374	0.11825	1.27396
2	1.27396	0.02706	1.57497	0.01718	1.25678
3	1.25678	0.00052	1.51408	0.00035	1.25643
4	1.25643	0.00000	1.51286	0.00000	1.25643

A real root of the given equation is 1.2564 correct to four decimal places.

Example.2. Find a real root of the equation $x\sin(x) + \cos(x) = 0$ correct to five(5) decimal places using Newton-Raphson.

Solution: The given equation is $x\sin(x) + \cos(x) = 0$

Let $f(x) = x\sin(x) + \cos(x)$, $\therefore f'(x) = x\cos(x) + \sin(x) - \sin(x) = x\cos(x)$

Then by Newton-Raphson iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad [f'(x_n) \neq 0], \quad n = 0, 1, 2, 3, \dots$$

This is we can write in the form as

$$x_{n+1} = x_n - h_n, \quad \text{where } h_n = \frac{f(x_n)}{f'(x_n)} \quad [f'(x_n) \neq 0], \quad n = 0, 1, 2, 3, \dots$$

Now $f(0) = 1$, $f(1) = 1.3818$, $f(2) = 1.4024$ and $f(3) = -0.5666$

Therefore a real root of the given equation lies in the interval (2,3).

Let us choose $x_0 = 2.5$

$$\text{Therefore } x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.5 - \frac{0.6950}{-2.0029} = 2.8470$$

Newton-Raphson iterative method tabulated as:

NI	x_n	$f(x_n)$	$f'(x_n)$	$h=f(x_n)/f'(x_n)$	x_{n+1}
0	2.500000	0.695037	-2.002859	-0.347022	2.847022
1	2.847022	-0.130355	-2.724392	0.047847	2.799175
2	2.799175	-0.002080	-2.636671	0.000789	2.798386
3	2.798386	-0.000001	-2.635186	0.000000	2.798386

A real root of the given equation is 2.79839 correct to five decimal places.

LECTURE 20:

Newton-Raphson Method for finding an assigned root of a positive real number.

Find the N^{th} . Root of a real number R, i.e. find the value of $\sqrt[N]{R}$.

Let $x = \sqrt[N]{R}$, $\therefore x^N = R$ or $x^N - R = 0$.

Let $f(x) = x^N - R$. $\therefore f'(x) = Nx^{N-1}$

Now by Newton-Raphson iterative formula we get,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad [f'(x_n) \neq 0], n = 0, 1, 2, 3, \dots$$
$$= x_n - \frac{x_n^N - R}{Nx_n^{N-1}} = \frac{Nx_n^N - x_n^N + R}{Nx_n^{N-1}} = \frac{(N-1)x_n^N + R}{Nx_n^{N-1}}, n = 0, 1, 2, 3, \dots$$

Example.3. Find the cube root of 35 correct up to four decimal places by using Newton-Raphson Method

Solution : Let $x = \sqrt[3]{35}$, $\therefore x^3 = 35$ or $x^3 - 35 = 0$, Here $N = 3$ and $R = 35$

Let initial approximation $x_0 = 3.5$

Therefore,

$$x_1 = \frac{(N-1)x_0^N + R}{Nx_0^{N-1}} = \frac{(3-1)x_0^3 + 35}{3x_0^{3-1}} = \frac{2(3.5)^3 + 35}{3(3.5)^2} = 3.2857143$$

$$x_2 = \frac{2(3.2857143)^3 + 35}{3(3.2857143)^2} = 3.2711315$$

$$x_3 = \frac{2(3.2711315)^3 + 35}{3(3.2711315)^2} = 3.271066$$

$$x_4 = \frac{2(3.271066)^3 + 35}{3(3.271066)^2} = 3.271066$$

Using Newton-Raphson Method, obtain iteration formula for the reciprocal of a number N and hence find the value of $\frac{1}{27}$, correct to four decimal places.

Solution: Let $x = \frac{1}{N}$

Then $\frac{1}{x} = N$, Therefore $\frac{1}{x} - N = 0$

Let $f(x) = \frac{1}{x} - N$, $\therefore f'(x) = -\frac{1}{x^2}$

Now by Newton-Raphson iterative formula we get,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad [f'(x_n) \neq 0], n = 0, 1, 2, 3, \dots$$

Then,

$$x_{n+1} = x_n - \frac{\frac{1}{27} - N}{-\frac{1}{x_n^2}} = x_n + \frac{(1 - Nx_n)}{x_n} x_n^2 = x_n + (1 - Nx_n)x_n$$

$$= (2 - Nx_n)x_n, \quad n = 0, 1, 2, 3, \dots$$

This is the required iterative formula.

Since the value of $\frac{1}{27}$ lies between $\frac{1}{30}$ and $\frac{1}{25}$ i.e. between 0.033 and 0.040.

For this problem let us take $N = 27$ and $x_0 = 0.03$

Therefore,

$$x_1 = (2 - Nx_0)x_0 = (2 - 27 \times 0.03) \times 0.03 = 0.0357$$

$$x_2 = (2 - Nx_1)x_1 = (2 - 27 \times 0.0357) \times 0.0357 = 0.0369888$$

$$x_3 = (2 - Nx_2)x_2 = (2 - 27 \times 0.0369888) \times 0.0369888 = 0.0370370$$

$$x_4 = (2 - Nx_3)x_3 = (2 - 27 \times 0.0370370) \times 0.0370370 = 0.0370370$$

Hence the value of $\frac{1}{27}$ is 0.0370 correct to four decimal places.

Module Number III:

Numerical Integration and Numerical Solution of Differential Equation

LECTURE 21:

Numerical Integration

It is, well known to us that, though $f(x)$ is completely known, it is not always easy, or rather, not always possible to evaluate a definite integral $\int_a^b f(x) dx$ by the known analytical methods of integration. For example $f(x) = e^{-x^5}, \frac{e^x}{|x+5|}$ etc. further, there are a number of situations where a table of values is known, viz., a table of population in different years etc. in these cases also, the known analytical integration methods fail to find a definite integral and the only numerical integration methods are suitable to evaluate a definite integral.

In this chapter we derive and analyses numerical methods to evaluate definite integrals of the form

$$I = \int_a^b f(x) dx$$

The principle of integration is: Fit up a suitable interpolation polynomial to the given set of values of $f(x)$ and then integrate it within the desired limits. For any interval $[a, b]$ by replacing the function $f(x)$ with a suitable polynomial $p(x)$ such that $\int_a^b p(x) dx$ is taken to be an approximation of the integral I . The approximation of I is usually known as “Numerical integration or Quadrature”.

Let $y=f(x)$ be a real valued function defined in $[a, b]$ such that the values of $f(x)$ are known for

$x=x_i$ ($i = 0, 1, 2, \dots, n$) whose all x_i lies in $[a, b]$ and $y_i=f(x_i)$, ($i = 0, 1, 2, \dots, n$). Also let $p(x)$ be the interpolating polynomial of degree at most n such that $p(x_i) = f(x_i)=y_i$

Thus, $p(x) \approx f(x)$ and so $I = \int_a^b f(x) dx \approx \int_a^b p(x) dx$

Then the expression $E(x) = \int_a^b f(x)dx - \int_a^b p(x)dx$ is known as the error of integration.

Degree of Precision.

A quadrature is said to have a degree of precision $M(>0)$ if it is exact but there is atleast one polynomial of degree $M+1$ for which it is not exact.

Composite rule

Sometimes it is more convenient to divide the interval $[a,b]$, of integration into finite number of sub-intervals, say $m(<n)$, $[a_{i-1}, a_i]$, $i=1,2,\dots,m$. Then we apply a quadrature formula separately to each of these sub-intervals and add the results. This formula is thus obtained is called *composite rule* corresponding to that *Quadrature* formula.

LECTURE 22:

Trapezoidal Rule

It is two points closed-typed Newton-Cotes quadrature formula in the interval $[a,b]$. Here $n=1$, and $h=b-a$. Thus, from (1) we get, replacing I by I_T

$$\begin{aligned} I_T &= (b-a) \sum_{i=0}^1 f(x_i) K_i^{(1)} = (b-a) [K_0^{(1)} f(x_0) + K_1^{(1)} f(x_1)] \\ &= (b-a) \left[\frac{1}{2} y_0 + \frac{1}{2} y_1 \right] \\ &= \frac{(b-a)}{2} [y_0 + y_1] \end{aligned}$$

The error is given by

$$E_T = \rho_2 = h^3 \int_0^1 \frac{t(t-1)}{2!} f''(\xi) dt = -\frac{h^3}{12} f''(\xi) = -\frac{(b-a)^3}{12} f''(\xi), (a < \xi < b)$$

The degree of precision in trapezoidal rule is 1.

Simpson's one-third rule

It is three points newton cotes quadrature formula in the interval $[a,b]$.

Here, $n=2$ and $h=(b-a)/2$, replacing I by I_s , we get from (1)

$$\begin{aligned} I_s &= (b-a) \sum_{i=0}^2 f(x_i) K_i^{(2)} = (b-a) [K_0^{(2)} f(x_0) + K_1^{(2)} f(x_1) + K_2^{(2)} f(x_2)] \\ &= \frac{(b-a)}{6} [y_0 + 4y_1 + y_2] \\ &= h/3 [y_0 + 4y_1 + y_2] \end{aligned}$$

The error is given by $E_s = \rho_3 = -\frac{h^5}{90} f''''(\xi), (a < \xi < b)$

The degree of precision in Simpson's 1/3 rule is 3.

LECTURE 23:

PROBLEM 1

Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by Trapezoidal and Simpson's 1/3 rule taking $n=6$.

Solution:

Let $f(x) = \frac{1}{1+x^2}$ since $n=6$, so $h=(1-0)/6=1/6$.

The tabulated values of $f(x)$ for different values of x are given below

x_i $i=0$ to 6	$Y_i=f(x_i)$ $i=0, 6$	$Y_i=f(x_i)$ $i=1,3,5$	$Y_i=f(x_i)$ $i=2,4$
0	1	-	-
1/6	-	0.973	-
2/6	-	-	0.9
3/6	-	0.8	-
4/6	-	-	0.6923
5/6	-	0.5902	-
1	0.5	-	-
	$\Sigma=1.5$	$\Sigma=2.3632$	$\Sigma=1.5923$

i) We have by composite Trapezoidal rule

$$\begin{aligned}\int_0^1 \frac{dx}{1+x^2} &\approx h/2[y_0+y_n+2(y_1+y_2+\dots+y_{n-1})] \\ &= 1/12[1.5+2(2.3632+1.5923)] \\ &= 0.7842\end{aligned}$$

ii) We have by composite Simpson's one third rule

$$\begin{aligned}\int_0^1 \frac{dx}{1+x^2} &\approx h/3[y_0+y_n+4(y_1+y_3+\dots)+2(y_2+y_4+\dots)] \\ &= 1/18[1.5+4(2.3632)+2(1.5923)] \\ &= 0.7854\end{aligned}$$

PROBLEM 2

Evaluate $\int_0^{\pi/2} \sqrt{\sin x} dx$ by Trapezoidal and Simpson's 1/3 rule taking $n=6$.

Solution- here $a=0$, $b=\pi/2$, $n=6$ $h= \pi/12$

X	0	$\pi/12$	$\pi/6$	$3\pi/12$	$4\pi/12$	$5\pi/12$	$\pi/2$
Y	0	0.5087	0.7071	0.8409	0.9306	0.9828	1.000

iii) We have by composite trapezoidal rule

$$\int_0^1 \frac{dx}{1+x^2} \approx h/2[y_0 + y_n + 2(y_1 + y_2 + \dots + y_{n-1})]$$

$$= \pi/24[1.0000 + 2(2.3324 + 1.6377)]$$

$$= 1.17027$$

ii) We have by composite Simpson's one third rule

$$\int_0^1 \frac{dx}{1+x^2} \approx h/3[y_0 + y_n + 4(y_1 + y_3 + \dots) + 2(y_2 + y_4 + \dots)]$$

$$= \pi/36[1.0000 + 4(2.3324) + 2(1.6377)]$$

$$= 1.18726$$

LECTURE 24:

Weddle's Rule

Let the values of a function $f(x)$ be tabulated at points x_i equally spaced by $h = x_{i+1} - x_i$, so $f_1 = f(x_1)$, $f_2 = f(x_2)$, Then Weddle's rule approximating the integral of $f(x)$ is given by the Newton-cotes -like formula

$$\int_{x_0}^{x_n} y dx = \left(\frac{3h}{10}\right) [y_0 + 2(y_6 + y_{12} + \dots y_{n-6}) + 5(y_1 + y_7 + \dots) + 6(y_3 + y_9 + \dots + y_{n-3}) + y_n]$$

Problem

Evaluate $\int_0^6 2x dx$ up to three decimal places, by Weddle's rule?

Solution:

Let $h = 1.0$. The tabulated values of x and y are given below

X	0	1	2	3	4	5	6
Y	0	2	4	6	8	10	12

From Weddle's rule, we have

$$\int_{x_0}^{x_n} y dx = \left(\frac{3h}{10}\right) [y_0 + 5(y_1 + y_5) + 6(y_3) + (y_2 + y_4) + y_6]$$

$$= 36$$

PROBLEM 2

Evaluate $\int_0^2 \frac{dx}{1+x^2}$ by Weddle's rule taking 12-intervals, correct upto 4 significant figures.

Solution:

Let $f(x) = \frac{1}{1+x^2}$ since $n=12$, so $h=(2-0)/12=1/6$.

The tabulated values of $f(x)$ for different values of x are given below

X	y
---	---

0	1.00000
1/6	0.97297
2/6	0.90000
3/6	0.80000
4/6	0.69231
5/6	0.59016
6/6	0.50000
7/6	0.42353
8/6	0.36000
9/6	0.30769
10/6	0.26470
11/6	0.22930
12/6	0.20000

From Weddle's rule, we have

$$I \approx (3h/10) [y_0 + y_{12} + 5(y_1 + y_5 + y_7 + y_{11}) + (y_2 + y_4 + y_8 + y_{10}) + 6(y_3 + y_9) + 2y_6]$$

$$= (3/60) \times 22.14295$$

$$= 1.071475$$

$$= 1.1071(\text{correct up to 4 D.p})$$

LECTURE 25:

Numerical Solution of Ordinary Differential Equation (ODE)

Taylor Series Method:

It is a simple and useful numerical method for solving an ordinary differential equation

$$y' = f(x, y), \quad y(x_0) = y_0$$

where f is a function of two variables x and y and (x_0, y_0) is a known point on the solution curve.

The existence of all higher order partial derivatives is assumed for y at $x = x_0$, then by Taylor series the value of y at any neighbouring point $x+h$ can be written as

$$y(x_0+h) = y(x_0) + h y'(x_0) + \frac{h^2}{2} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \dots$$

where ' represents the derivative with respect to x . Since at x_0, y_0 is known, y' at x_0 can be found by computing $f(x_0, y_0)$. Similarly higher derivatives of y at x_0 also can be computed by making use of the relation

$$y' = f(x, y)$$

$$y'' = f_x + f_y y'$$

$$y''' = f_{xx} + 2f_{xy} y' + f_{yy} y'^2 + f_y y''$$

and so on.

$$\text{Then } y(x_0+h) = y(x_0) + h f + h^2 (f_x + f_y y') / 2! + h^3 (f_{xx} + 2f_{xy}y' + f_{yy} y'^2 + f_{yy} y'') / 3! + o(h^4)$$

Hence the value of y at any neighboring point $x_0 + h$ can be obtained by summing the above infinite series. However, in any practical computation, the summation has to be terminated after some finite number of terms. If the series has been terminated after the p^{th} derivative term then the approximated formula is called the Taylor series approximation to y of order p and the error is of order $p+1$. The same can be repeated to obtain y at other points of x in the interval $[x_0, x_n]$ in a marching process.

Error in the approximation :

The Taylor series method of order p has the property that the final global error is of order $O(h^{p+1})$; hence p can be chosen as large as necessary to make the error as small as desired. If the order p is fixed, it is theoretically possible to a priori determine the size of h so that the final global error will be as small as desired. Since

$$E_p = \frac{1}{(p+1)!} h^{p+1} y^{p+1}(x + \theta h), \quad 0 < \theta < 1$$

Making use of finite differences, the $(p+1)^{\text{th}}$ derivative of y at $x + \theta h$ can be approximated as

$$E_p = \frac{h^p (y^p(x + \theta h) - y^p(x))}{(p+1)!}$$

However, in practice one usually computes two sets of approximations using step sizes h and $h/2$ and compares the solutions.

For $p = 4$, $E_4 = c * h^4$ and the same with step size $h/2$, $E_4 = c * (h/2)^4$, that is if the step size is halved the error is reduced by an order of $1/16$.

EXAMPLE

Using Taylor series method, find $y(0.1)$ for $y' = x - y^2$, $y(0) = 1$ correct upto four decimal places.

Solution:

$$\text{Given } y' = f(x, y) = x - y^2$$

$$\Rightarrow y'' = 1 - 2yy', \quad y''' = -2yy'' - 2y'^2, \quad y^{iv} = -2yy''' - 6y'y'', \quad y^v = -2yy^{iv} - 8y'y''' - 6y''^2$$

Since at $x = 0$, $y = 1$;

$$y' = -1, \quad y'' = 3, \quad y''' = -8, \quad y^{iv} = 34 \quad \text{and} \quad y^v = -186$$

The forth order Taylor's formula is

$$y(x) = y(x_0) + (x-x_0) y'(x_0, y_0) + (x-x_0)^2 y''(x_0, y_0)/2! + (x-x_0)^3 y'''(x_0, y_0)/3! + (x-x_0)^4 y^{iv}(x_0, y_0)/4! + h^5 y^v(x_0, y_0)/5! + \dots$$

$$= 1 - x + 3 x^2/2! - 8 x^3/3! + 34 x^4/4! - 186 x^5/5! \quad (\text{since } x_0 = 0)$$

$$= 1 - x + 3x^2/2 - 4x^3/3 + 17x^4/12 - 31x^5/20$$

Now

$$\begin{aligned} y(0.1) &= 1 - (0.1) + 3(0.1)^2/2 - 4(0.1)^3/3 + 17(0.1)^4/12 - 31(0.1)^5/20 \\ &= 0.9 + 3(0.1)^2/2 - 4(0.1)^3/3 + 17(0.1)^4/12 - 31(0.1)^5/20 \\ &= 0.915 - 4(0.1)^3/3 + 17(0.1)^4/12 - 31(0.1)^5/20 \\ &= 0.9137 + 17(0.1)^4/12 - 31(0.1)^5/20 \\ &= 0.9138 - 31(0.1)^5/20 \\ &= 0.9138 \end{aligned}$$

Since the value of the last term does not add upto first four decimal places, the Taylor series formula of order four is sufficient to find **y(0.1)** accurate upto four decimal places.

LECTURE 26:

Euler's Method:

Working formula for the ODE $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$, where x_0 is the initial value of x and y_0 is the initial value of y .

$x_1 = x_0 + h$, where h is the step length, now approximate value of y at $x = x_1 (= x_0 + h)$ written as

$$y(x_1) = y_1 = y_0 + hf(x_0, y_0).$$

Again the approximate value of y at $x = x_2 (= x_1 + h)$ written as

$$y(x_2) = y_2 = y_1 + hf(x_1, y_1).$$

Then the general formula we written as

$$y(x_{n+1}) = y_{n+1} = y_n + hf(x_n, y_n) \quad (n = 0, 1, 2, 3, \dots) \quad \dots \dots \dots (.1)$$

Where $x_n = x_0 + nh$

Example.1. Find $y(0.2)$ for the ordinary differential equation (ODE) $\frac{dy}{dx} = x + y$, with the initial condition $y=1$ when $x=0$, correct up to 4 decimal places, taking step length $h=0.05$ by Euler's Method.

Solution : Here $x_0 = 0$, $y_0 = 1$, $h=0.05$ and $\frac{dy}{dx} = f(x, y) = x + y$

Using the Euler's successive approximations formula, we get,

$$x_1 = x_0 + h = 0 + 0.05 = 0.05$$

$$y_1 = y(x_1) = y(0.05) = y(0) + 0.05[0 + 1] = 1 + 0.05 = 1.05$$

$$x_2 = x_1 + h = 0.05 + 0.05 = 0.10$$

$$\begin{aligned} y_2 = y(x_2) = y(0.1) &= y(0.05) + 0.05[x_1 + y_1] = 1.05 + 0.05[0.05 + 1.05] \\ &= 1.05 + 0.05(1.1) = 1.05 + 0.0550 = 1.1050 \end{aligned}$$

$$x_3 = x_2 + h = 0.10 + 0.05 = 0.15$$

$$\begin{aligned} y_3 = y(x_3) = y(0.15) &= y(0.10) + 0.05[x_2 + y_2] = 1.1050 + 0.05[0.10 + 1.1050] \\ &= 1.1050 + 0.05(1.2050) = 1.1050 + 0.0603 = 1.1653 \end{aligned}$$

$$x_4 = x_3 + h = 0.15 + 0.05 = 0.20$$

$$\begin{aligned} y_4 = y(x_4) = y(0.20) &= y(0.15) + 0.05[x_3 + y_3] = 1.1653 + 0.05[0.15 + 1.1653] \\ &= 1.1653 + 0.05(1.3153) = 1.1653 + 0.0658 = 1.2311 \end{aligned}$$

Hence $y(0.2) = 1.2311$ correct to four(4) decimal places.

Table of Example 1:

NI	x_0	y_0	$f(x_0, y_0)$	y_1	$x_1 = x_0 + h$
0	0.0000	1.000000	+1.000000	1.050000	0.0500
1	0.0500	1.050000	+1.100000	1.105000	0.1000
2	0.1000	1.105000	+1.205000	1.165250	0.1500
3	0.1500	1.165250	+1.315250	1.231013	0.2000

Hence the value of $y(0.20) = 1.2310$

Example 2. Find $y(4.4)$ for the ordinary differential equation (ODE) $\frac{dy}{dx} = \frac{2-y^2}{5x}$, with the initial condition $y=1$ when $x=4$, correct up to 4 decimal places, taking step length $h=0.2$, by Euler's Method.

Solution : Here $x_0 = 4$, $y_0 = 1$, $h=0.2$ and $\frac{dy}{dx} = f(x, y) = \frac{2-y^2}{5x}$

Using the Euler's successive approximations formula, we get, $x_1 = x_0 + h = 4 + 0.2 = 4.2$
 $y_1 = y(x_1) = y(4.2) = y(4) + 0.2[(2.0 - 1 \times 1)/(5 \times 4)] = 1 + 0.2(0.05000) = 1.0100$

$x_2 = x_1 + h = 4.2 + 0.2 = 4.4$

$y_2 = y(x_2) = y(4.4) = y(4.2) + 0.2[(2.0 - 1.01 \times 1.01)/(5 \times 4.2)] = 1.01 + 0.2(0.0467) = 1.0193$

Hence $y(4.4) = 1.0193$ correct to four(4) decimal places.

Table of Example 2:

NI	x_0	y_0	$f(x_0, y_0)$	y_1	$x_1 = x_0 + h$
0	4.0000	1.000000	+0.050000	1.010000	4.2000
1	4.2000	1.010000	+0.046662	1.019332	4.4000

Hence the value of $y(4.40) = 1.0193$

LECTURE 27:

Euler's Modified Method:

Euler's Modified Method:

Working formula for the ODE $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$, where x_0 is the initial value of x and y_0 is

the initial value of y . To minimize the error we use Euler's Modified Method instead of Euler's Method. To get the value of y at $x_1 = x_0 + h$, where h is the step length. Here we are using two formulas:

1) One is Euler's Method (as we learn earlier is known as Predictor formula), approximate value of y at $x = x_1$ ($=x_0+h$) written as

$$y(x_1) = y_1^{(0)} = y_0 + hf(x_0, y_0) \dots \dots \dots (2)$$

And to get the second approximation for y_1 we replace $f(x_0, y_0)$ by average of $f(x_0, y_0)$ and $f(x_1, y_1^{(0)})$

2) Another one is second approximation for y_1 (is known as Corrector formula) at $x = x_1$ ($=x_0+h$) written as

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})] \dots \dots \dots (.3)$$

Similarly, we get the third approximation for y1 as

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})] \dots \dots \dots (.4)$$

In general we can write 2nd ., 3rd ., 4th as

$$y_n^k = y_{n-1} + \frac{h}{2}[f(x_{n-1}, y_{n-1}) + f(x_n, y_n^{(k-1)})] \quad [(n=1, 2, 3, \dots); \text{for particular } n (k=1, 2, 3, \dots)]$$

is used to approximate y_n.

Note : Euler's Modified Method is also known as Predictor-Corrector Method.

Example 3: Evaluate y(1.2) for the ODE $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$ with y(1) = 1 by Euler's Modified Method correct up to 4 decimal places.

Solution: Here $x_0 = 1$, $y_0 = 1$, and $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$, therefore $\frac{dy}{dx} = f(x, y) = \frac{1}{x^2} - \frac{y}{x} = \frac{1-xy}{x^2}$

Let $h=0.1$, so that $x_1 = x_0 + h = 1+0.1 = 1.1$

$$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.1[(1-1)/(1 \times 1)] = 1 + 0 = 1$$

Now from (.3) we get

$$\begin{aligned} y_1^{(1)} &= y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})] = 1 + 0.1[(1-1)/(1 \times 1) + (1-1.1 \times 1)/(1.1 \times 1.1)]/2 \\ &= 1 + 0.1[0 - 0.082645]/2 = 1 - 0.0041322 = 0.99587 \end{aligned}$$

$$\begin{aligned} y_1^{(2)} &= y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})] = 1 + 0.1[0 + (1-1.1 \times 0.99587)/(1.1 \times 1.1)]/2 \\ &= 1 + 0.05(-0.07889) = 0.99606 \end{aligned}$$

$$y_1^{(3)} = 1 + 0.1[0 + (1-1.1 \times 0.99606)/(1.1 \times 1.1)]/2 = 1 + 0.05(-0.079063) = 0.99605$$

$$y_1^{(4)} = 1 + 0.1[0 + (1-1.1 \times 0.99605)/(1.1 \times 1.1)]/2 = 1 + 0.05(-0.079054) = 0.99605$$

Hence $y_1 = y(1.1) = 0.99605 \approx 0.9961$ (Approximately correct to 4 decimal places)

Now $x_2 = x_1 + h = 1.1 + 0.1 = 1.2$ and $f(x_1, y_1) = (1 - 1.1 \times 0.9961)/(1.1 \times 1.1) = -0.07910$

$$y_2^{(0)} = y_1 + hf(x_1, y_1) = 0.9961 + 0.1(-0.07910) = 0.98819$$

$$\begin{aligned} y_2^{(1)} &= y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(0)})] = 0.9961 + 0.05[-0.07910 + (1 - 1.2 \times 0.98819)/(1.2 \times 1.2)] \\ &= 0.9961 + 0.05(-0.07910 - 0.12905) = 0.98569 \end{aligned}$$

$$\begin{aligned} y_2^{(2)} &= y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})] = 0.9961 + 0.05[-0.07910 + (1 - 1.2 \times 0.98569)/(1.2 \times 1.2)] \\ &= 0.9961 + 0.05(-0.07910 - 0.12696) = 0.9858 \end{aligned}$$

$$\begin{aligned} y_2^{(3)} &= y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(2)})] = 0.9961 + 0.05[-0.07910 + (1 - 1.2 \times 0.9858)/(1.2 \times 1.2)] \\ &= 0.9961 + 0.05(-0.07910 - 0.12706) = 0.98579 \end{aligned}$$

Hence $y_2 \approx 0.9858$ (correct to 4 decimal places)

$\therefore y(1.2) \approx 0.9858$

NI	x_0	y_0	y_1	y_2	$x_1=x_0+h$
0	1.0000	1.00000	1.00000 1.00000 0.99587 0.99606 0.99605	0.99587 0.99606 0.99605	1.1000 1.1000 1.1000 1.1000 1.1000
1	1.1000	0.99605	0.98814 0.98814 0.98564 0.98575 0.98574	0.98564 0.98575 0.98574	1.2000 1.2000 1.2000 1.2000 1.2000

Hence $y(1.20)=0.98574$

LECTURE 28:

Example 4: Evaluate $y(0.02)$ for the ODE $\frac{dy}{dx} = x^2 + y$ with $y(1) = 0$ by Euler's Modified Method correct up to 4 decimal places, by taking step length $h=0.01$.

Solution: Here $x_0 = 0$, $y_0 = 1$, and $\frac{dy}{dx} = x^2 + y$, therefore $\frac{dy}{dx} = f(x, y) = x^2 + y$

$h=0.01$, so that $x_1 = x_0 + h = 0+0.01 = 0.01$

$$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.01(0x_0 + 1) = 1 + 0.01 = 1.01$$

Now from (.3) we get

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})] = 1 + 0.01[(0x_0 + 1) + (0.01x_0.01 + 1.01)]/2$$

$$= 1 + 0.005[1 + 0.001 + 1.01] = 1 + 0.010055 = 1.010055$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})] = 1 + 0.005[(0x_0 + 1) + (0.01x_0.01 + 1.010055)]$$

$$= 1 + 0.005(1 + 1.010155) = 1.010051$$

$$y_1^{(3)} = 1 + 0.005[(0x_0 + 1) + (0.01x_0.01 + 1.010051)] = 1 + 0.005(1 + 1.010151) = 1.010051$$

Hence $y_1 = y(0.01) = 1.010051$

Now $x_2 = x_1 + h = 0.01 + 0.01 = 0.02$ and $f(x_1, y_1) = (0.01x_0.01 + 1.010051) = 1.010151$

$$y_2^{(0)} = y_1 + hf(x_1, y_1) = 1.010051 + 0.01x_1.010151 = 1.0201525 \approx 1.02015$$

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(0)})] = 1.010051 + 0.005[1.010151 + (0.02x_0.02 + 1.020150)]$$

$$= 1.010051 + 0.010153505 \approx 1.020205$$

$$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})] = 1.010051 + 0.005[1.010151 + (0.02x_0.02 + 1.020205)]$$

$$= 1.010051 + 0.01015378 \approx 1.02025$$

Hence $y_2 \approx 1.02025$ (correct to 5 decimal places)

$\therefore y(0.02) \approx 1.02025$

NI	x_0	y_0	y_1	y_2	$x_1=x_0+h$
0	0.0000	1.00000	1.01000 1.01000 1.01005	1.01005 1.01005	0.0100 0.0100 0.0100

1	0.0100	1.01005	1.02015		0.0200
			1.02015	1.02020	0.0200
			1.02020	1.02020	0.0200

Hence $y(0.02)=1.02020$

LECTURE 29:

Predictor-Corrector method:

To find more accurately numerical solution of an ordinary differential equation we use **Predictor-Corrector method**. Firstly we use Predictor formula to get approximate value of $y_{n+1} = y(x_{n+1})$, then to improve the approximate value, we use Corrector formula. Corrector formula is more accurate than the predictor formula. Predictor-Corrector method is a multi-step method.

Milne's method: This multi-step method obtained by Milne by integrating over more than one step.

$$\frac{dy}{dx} = f(x, y) = f \text{ and } f(x_n, y_n) = f_n \text{ with}$$

$$y(x_n) = y_n, y(x_{n-1}) = y_{n-1}, y(x_{n-2}) = y_{n-2}, y(x_{n-3}) = y_{n-3}$$

Milne's predictor formula of order 4 is

$$y_{n+1} = y_{n-3} + \frac{4h}{3}(2f_{n-2} - f_{n-1} + 2f_n) \quad \text{--- --- --- --- (1)}$$

which is the 4-step explicit recursion formula of order four(4).

Milne's corrector formula of order 4 is

$$y_{n+1} = y_{n-1} + \frac{h}{3}(f_{n-1} + 4f_n + f_{n+1}) \quad \text{--- --- --- --- (2)}$$

which is 2-step implicit recursion formula of order four(4).

By using (1) and (2) we have to find out y_{n+1} where x_{n+1} is given.

LECTURE 30:

Example 5: Find the value of $y(0.4)$ for the Ordinary Differential Equation $\frac{dy}{dx} = xy + y^2$, with $y(0) = 1, y(0.1)=1.1169, y(0.2)=1.2773, y(0.3)=1.5040$ correct to three decimal places.

Solution: For the given problem $f(x,y) = xy + y^2$ and also given that $x_0=0.0, x_1=0.1, x_2=0.2, x_3=0.3, h=0.1$ and $y_0=1.0, y_1=1.1169, y_2=1.2773, y_3=1.5040$

Therefore $f_0=f(x_0,y_0)=f(0.0,1)=1, f_1=f(x_1,y_1)=f(0.1,1.1169)=0.1 \times 1.1169 + (1.1169)^2 = 1.359156$

$$f_2=f(x_2,y_2)=f(0.2,1.2773)=0.2 \times 1.2773 + (1.2773)^2 = 1.886955$$

$$f_3=f(x_3,y_3)=f(0.3,1.5040)=0.3 \times 1.5040 + (1.5040)^2 = 2.713216$$

Now putting $n=3$ in equation (1) and (2) we get as

$$y_4^p = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) \quad \text{--- --- --- --- (3)}$$

$$y_4^c = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \quad \text{--- --- --- --- (4)}$$

Now from (3) we get,

$$y_4^p = 1 + \frac{4 \times 0.1}{3}(2 \times 1.359156 - 1.886955 + 2 \times 2.713216) = 1.834372$$

$$\text{Hence } f_4^p = f(x_4, y_4^p) = 0.4 \times 1.834372 + (1.834372)^2 = 4.098669$$

$$\therefore y_4^{c1} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^p) = 1.2773 + \frac{0.1}{3}(1.886955 + 4 \times 2.713216 + 4.098669) = 1.838583$$

$$\therefore \text{Again } f_4^{c1} = f(x_4, y_4^{c1}) = 0.4 \times 1.838583 + (1.838583)^2 = 4.115821$$

$$\text{Hence } y_4^{c2} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^{c1}) = 1.2773 + \frac{0.1}{3}(1.886955 + 4 \times 2.713216 + 4.115821) = 1.839155$$

$$\therefore \text{Again } f_4^{c2} = f(x_4, y_4^{c2}) = 0.4 \times 1.839155 + (1.839155)^2 = 4.118153$$

$$\text{Hence } y_4^{c3} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^{c2}) = 1.2773 + \frac{0.1}{3}(1.886955 + 4 \times 2.713216 + 4.118153) = 1.839232$$

$$\therefore \text{Again } f_4^{c3} = f(x_4, y_4^{c3}) = 0.4 \times 1.839232 + (1.839232)^2 = 4.118467$$

$$\text{Hence } y_4^{c4} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^{c3}) = 1.2773 + \frac{0.1}{3}(1.886955 + 4 \times 2.713216 + 4.118467) = 1.839243$$

$$\therefore y_4^{c3} = y_4^{c4} = 1.8392, \text{ correct up to four(4) decimal places.}$$

$$\text{Hence } y(0.4) = 1.8392 \text{ correct up to four(4) decimal places.}$$

LECTURE 31:

Runge-Kutta Methods:

Runge-Kutta Methods are used to finding numerical solution of an ordinary differential equation. Runge-Kutta Methods gives us more accurate result than Euler's and Euler's Modified Method. Here we discussed two methods as:

- Second Order Runge-Kutta Method:
- Fourth Order Runge-Kutta Method:

Fourth Order Runge-Kutta Method:

Computational formula for an ordinary differential equation $y' = \frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$ is

$y_1 = y_0 + k$ where $k = (k_1 + k_2)/2$ and $k_1 = h.f(x_0, y_0)$; $k_2 = h.f(x_0 + h, y_0 + k_1)$. The error in this formula is $O(h^3)$.

Example 1: Compute $y(0.2)$ for the ordinary differential equation $\frac{dy}{dx} = x - y$ with $y(0) = 1$, taking $h = 0.1$, by Runge-Kutta Method of order 4, correct to six decimal places.

Solution: For the given problem Here $x_0 = 0$, $y_0 = 1$, and $\frac{dy}{dx} = x - y$, therefore $f(x, y) = x - y$ and $h = 0.1$.

now for $y(0.1)$

$$k_1 = hf(x_0, y_0) = 0.1(0 - 1) = -0.1$$

$$k_2 = hf(x_0 + h/2, y_0 + k_1/2) = 0.1[(0 + 0.05) - (1 - 0.05)] = 0.1(-0.9) = -0.09$$

$$k_3 = hf(x_0 + h/2, y_0 + k_2/2) = 0.1[(0 + 0.05) - (1 - 0.045)] = 0.1(1 - 0.095) = 0.1(-0.905) = -0.0905$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1[(0 + 0.1) - (1 - 0.0905)] = 0.1(1 - 0.1905) = 0.1(-0.8095) = -0.08095$$

$$k = (k_1 + 2k_2 + 2k_3 + k_4)/6 = [-0.1 + 2(-0.09) + 2(-0.0905) - 0.08095] = (-0.54195)/6 = -0.090325$$

$$y_1 = y_0 + k = 1 - 0.090325 = 0.909675$$

$$\therefore y(0.1) = y_1 = 0.909675$$

Now for $y(0.2)$: $x_1 = 0.1$ and $y_1 = 0.909675$

$$k_1 = hf(x_1, y_1) = 0.1f(0.1, 0.909675) = 0.1(0.1 - 0.909675) = -0.0809675$$

$$\begin{aligned}
k_2 &= hf(x_1+h/2, y_1+k_1/2) = 0.1f[(0.1+0.05), (0.909675-0.04048375)] = 0.1f(0.15, 0.86919125) \\
&= 0.1(0.15 - 0.86919125) = -0.0719191 \\
k_3 &= hf(x_1+h/2, y_1+k_2/2) = 0.1f[(0.1+0.05), (0.909675-0.03595955)] = 0.1f(0.15, 0.87371545) \\
&= 0.1(0.15 - 0.87371545) = -0.0723715 \\
k_4 &= hf(x_1+h, y_1+k_3) = 0.1f(0.1+0.1, 0.909675-0.0723715) = 0.1f(0.2, 0.8373035) \\
&= 0.1(0.2 - 0.8373035) = -0.0637303 \\
k &= (k_1 + 2k_2 + 2k_3 + k_4)/6 = [-0.0809675 + 2(-0.0719191) + 2(-0.0723715) - 0.0637303] \\
&= (-0.433279)/6 = -0.07221317 \\
y_2 &= y_1 + k = 0.909675 - 0.07221317 = 0.837462 \text{ (correct to 6 decimal places)} \\
\therefore y(0.2) &= y_2 = 0.837462
\end{aligned}$$

LECTURE 32:

Numerical Solution of Partial Differential Equations (2L)

The **general form** of a linear PDE in 2 independent variables is

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G$$

The coefficients A, B, C, *etc.* may be constants or variables.

The **order** of a pde is the order of the highest partial derivative present.

For example, $u_t = u_{xx}$ is second order.

In the general form above, $u(x,y)$ is the **dependent variable** and x and y are the **independent variables**.

Pde's are **linear** if no products of the dependent variable or its derivatives, or functions such as sine involving the dependent variable, are present. Otherwise the pde is **nonlinear**. Generally, linear pde's are much easier to solve, except for particular nonlinear pdes where analytic solutions are known.

Pde's may be **classified** according to whether they are:

Hyperbolic $B^2 > 4AC$ Vibrating systems, waves, *etc.*

Parabolic $B^2 = 4AC$ Heat flow, diffusion, *etc.*

Elliptic $B^2 < 4AC$ Equilibrium problems, electrostatics, *etc.*

Finite Difference Approximations(FDA)

Taylor Series Expansion Approximations to first and second derivatives of $f(x)$

The Taylor series expansion for a function of one variable about the point x is

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + O(h^3)$$

The notation $O(h^3)$ indicates that the series, when truncated at the quadratic term in h , contains errors that scale as h^3 and higher powers of h .

We can immediately obtain an approximation to the derivative of $f(x)$ from the first two terms of the expansion.

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h^1)$$

Note that, even though we neglect terms of $O(h^2)$ in the expansion, since we divide through by h to obtain the derivative expression the approximation is correct to $O(h^1)$ only. Note that there is an **asymmetry** in this approximation to the derivative at x , since the function at x and $x+h$ occur, but not the function at $x-h$. This is therefore referred to as a **forward difference approximation**. It is possible to expand $f(x)$ in the negative direction in the Taylor expansion and hence to obtain a **backward difference approximation**

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) + O(h^3)$$

$$f'(x) = \frac{f(x) - f(x-h)}{h} + O(h^1)$$

By combining forward and backward difference approximations it is possible to obtain a **central difference approximation** to the derivative of $f(x)$ at x that contains errors of order $O(h^2)$. Note the difference in scaling of errors for the central difference approximation when compared with the forward and backward difference approximations.

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + O(h^3)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) + O(h^3)$$

$$f(x+h) - f(x-h) = 2hf'(x) + 2\frac{h^3}{3!}f'''(x) + O(h^4)$$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$

By retaining terms in the Taylor series to order h^3 we can obtain an approximation for the second derivative which contains errors of order h^2

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + O(h^4)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + O(h^4)$$

add these expansions to obtain

$$h^2f''(x) = f(x+h) - 2f(x) + f(x-h) + O(h^4)$$

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2)$$

LECTURE 33:

CRANK – NICHOLSON DIFFERENCE METHOD

Let us solve the parabolic equation

$$\frac{\partial^2 u}{\partial x^2} = a \frac{\partial u}{\partial t} \quad \text{---- (1)}$$

With boundary conditions

$$u(0,t) = T_0$$

$$u(l,t) = T_1$$

and the initial condition $u(x, 0) = f(x)$

At $u_{i,j}$
$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

and at $u_{i,j+1}$
$$u_{xx} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}$$

Taking the average of these two values

$$u_{xx} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{2h^2}$$

Using
$$u_t = \frac{u_{i,j+1} - u_{i,j}}{k}$$

Equation (1) reduces to

$$\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{2h^2} = a \frac{u_{i,j+1} - u_{i,j}}{k}$$

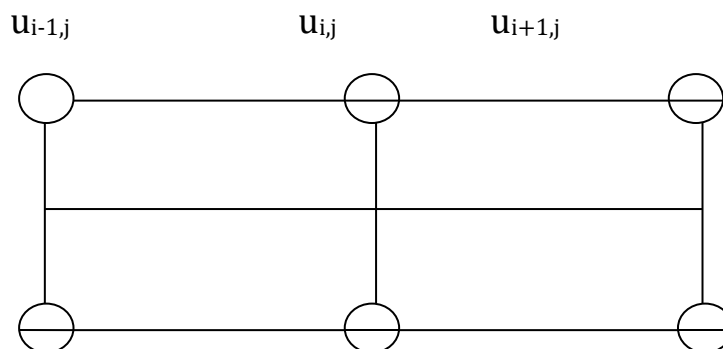
Setting $\frac{k}{ah^2} = \lambda$, the above reduces as

$$\frac{1}{2} \lambda u_{i+1,j+1} + \frac{1}{2} \lambda u_{i-1,j+1} - (\lambda + 1)u_{i,j+1} = \frac{1}{2} \lambda u_{i+1,j} - \frac{1}{2} \lambda u_{i-1,j} + (\lambda - 1)u_{i,j}$$

$$\text{or, } \lambda (u_{i+1,j} + u_{i-1,j}) - 2(\lambda + 1)u_{i,j} = 2(\lambda - 1) u_{i,j} - \lambda (u_{i+1,j+1} + u_{i-1,j+1}) \quad \text{---- (2)}$$

Useful notes

(1) The six points in the above formula are shown below



$$u_{i-1,j+1}$$

$$u_{i,j+1}$$

$$u_{i+1,j+1}$$

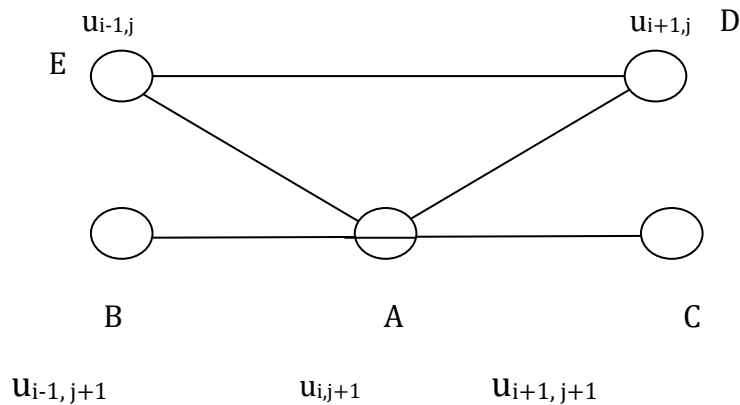
j – the root 't' (1st row)

j+1 – the root of 't' (3rd row)

- (2) A convenient choice of 1 makes the formula (2) simple. Setting $\lambda = 1$ i.e. ($k = ah^2$), the Crank – Nicholson formula reduces as

$$u_{i,j+1} = \frac{1}{4}[u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}] \quad \text{---- (3)}$$

In examples, we will use this simplified formula (3) subject to $k = ah^2$



The value of u at A = average of the values at B, C, D, E

- (3) The Crank-Nicholson scheme converges for all values of λ .

Example 1: Using Crank-Nicholson's method, solve

$$U_{xx} = 16 u_t, \quad 0 < x < 1, \quad t > 0, \quad \text{given}$$

$$U(x,0) = 0, \quad u(0,t) = 0, \quad u(1,t) = 100t$$

Compute 'u' for one step in 't' direction taking $h = \frac{1}{4}$

Solution: Here, $a = 16$, $h = \frac{1}{4}$

$$K = ah^2 = 16 \cdot (\frac{1}{4})^2$$

By Crank – Nicholson's formula

$$u_{i,j+1} = \frac{1}{4} [u_{i,j+1} + u_{i-j,j+1} + u_{i+1,j} + u_{i-1,j}] \quad \text{.....(1)}$$

i \ j	0	0.25	0.50	0.75	1
0	0	0	0	0	0

1	0	u_1	u_2	u_3	100
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Using equation (1), we get,

$$\Rightarrow u_1 = \frac{1}{4} (0 + 0 + 0 + u_2) \quad \dots\dots\dots(2)$$

$$\Rightarrow u_2 = \frac{1}{4} (u_3 + u_1 + 0 + 0) \quad \dots\dots\dots(3)$$

$$\Rightarrow u_3 = \frac{1}{4} (100 + u_2 + 0 + 0) \quad \dots\dots\dots(4)$$

Solving (2), (3) and (4), we get

$$u_1 = 1.7857, \quad u_2 = 7.1429 \quad u_3 = 26.7857$$

MCQ NUMERICAL METHODS:

If $y=f(x)$ be defined at $(n+1)$ distinct interpolating points, then the degree of the Lagrangin function is

- a) at least n b) at most n c) $n+1$ d) n

The order of convergence of the Newton-Raphson's method is

- a) 0 b) 1 c) 2 d) 3 .

When Gauss-Jordan method is used to solve the system of equation which can be written in the form $AX=b$, A is transform to a

- a) Lower triangular matrix b) Upper triangular matrix
c) Diagonal identity matrix d) Symmetric matrix

If $dy/dx = xy$ and $y(0) = 2$, then k_2 according to Runge-Kutta method is $[h = 0.2]$

- a) 0.4 b) 0.04 c) 0.004 d) 0.0004

The order of convergence of Newton Raphson method is

- a) 3 b) 2 c) 1 d) 4

The degree of precession of simpson's $1/3$ rule is

- a) 3 b) 2 c) 1 d) 4

The first order forward difference of constant function is

- a) 0 b) 2 c) 1 d) none of these

Newton's forward interpolation formula is used for

- a) equispaced nodes b) unequispaced nodes
c) both (a) and (b) d) none of these

Newton Raphson method is also known as

- a) straight line b) tangent c) normal d) cyclic

The relation between Δ and ∇ is

- a) $\Delta \nabla = \Delta - \nabla$ b) $\Delta \nabla = I$ c) $\Delta = I / \nabla$ d) none of these

Correct to 5 decimal places, 1.004355 is rounded to

- a) 1.00436 b) 1.00435 c) 1.004355 d) none of these

Which is the direct method?

- a) Gauss elimination method b) Gauss Jacobi method
c) Gauss Seidel method d) none of these

Modified Euler method has a truncation error of order

- a) 0 b) 2 c) 1 d) none of these

Which of the following is not a computational error?

- a) truncation error b) round off error c) inherent error d) none of these

Runge Kutta 4th order method has a truncation error of order

- a) 4 b) 5 c) 3 d) none of these

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